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## Non Interpolation in Morrey-Campanato and Block Spaces

OSCAR BLASCO – ALBERTO RUIZ – LUIS VEGA

**Abstract.** We prove non interpolation results for the family of Morrey spaces. We introduce a scale of block spaces, which are preduals of Morrey spaces in some range. Negative interpolation results are also obtained in this case.

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### 1. – Introduction

The spaces  $\mathcal{L}^{p,\alpha}$ , for the range  $\alpha \in (0, n/p]$  and  $p \in [1, \infty]$ , were introduced by Morrey in order to study regularity questions which appear in the Calculus of Variations, later Campanato extended the definition to the range  $\alpha \in (-1, n/p]$ .

$\mathcal{L}^{p,\alpha}$  is defined as the set of functions  $f$  locally in  $L^p(\mathbb{R}^n)$  and such that there exists a constant  $\sigma$  for which

$$(1.1) \quad \sup_Q r^\alpha \left( r^{-n} \int_Q |f(x) - \sigma|^p dx \right)^{1/p} < \infty,$$

where the sup is taken over all the cubes in  $\mathbb{R}^n$  and  $r$  denotes the side length. The norm  $\|\cdot\|_{p,\alpha}$  is defined as the infimum of (1.1) when  $\sigma \in \mathbb{R}$ .

In the range defined by Morrey, functions in this space have been used as weights, to substitute the Lebesgue spaces  $L^p$  by weighted- $L^2$ , in Sobolev-Poincaré inequalities, unique continuation, potentials in wave and Schrödinger equations and some other problems in PDE, see [CS], [ChR], [FP], [Sc], [T], [W]. There are still some interesting open problems, for example in unique continuation and in the restriction properties of the Fourier transform, see [K], [RV1], [RV2]. In this range we can, without loss of generality, take  $\sigma = 0$ , the

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endpoint case  $p = \frac{n}{\alpha}$  is just  $L^p$ , being in the other case,  $p < n/\alpha$ ,  $L^p$  strictly included in  $\mathcal{L}^{p,\alpha}$ .

When  $\alpha < 0$  (Campanato's extended range) it has been proved that  $\mathcal{L}^{p,\alpha}$  is the space of  $(-\alpha)$ -Hölder continuous functions, see [C] and [M]. When  $\alpha = 0$  we have BMO.

In this work we reduce ourselves to the range  $\alpha \in (0, n/p]$ ,  $p \in (1, \infty]$ . Therefore (see, for instance [Ku]) we have  $\mathcal{L}^{p,\alpha}$  is the set of functions  $f$  locally in  $L^p(\mathbb{R}^n)$  and such that

$$(1.2) \quad \|f\|_{p,\alpha} = \sup_Q r^\alpha \left( r^{-n} \int_Q |f(x)|^p dx \right)^{1/p} < \infty,$$

where the sup is taken over all the cubes in  $\mathbb{R}^n$  and  $r$  denotes the side length. Our concerns are duality and interpolation properties of this two-parameters family of spaces. A few more historical comments are in order.

Interpolation properties of  $\mathcal{L}^{p,\alpha}$  were the objects of attention in several works during the 60's. Stampacchia [St], and Campanato and Murthy [CM] proved that if  $T$  is a linear operator bounded from  $L^{q_i}$  to  $\mathcal{L}^{p_i,\alpha_i}$ ,  $i = 1, 2$ , with operator norm  $K_i$ , then  $T$  is bounded from  $L^{q_\theta}$  to  $\mathcal{L}^{p_\theta,\alpha_\theta}$  with norm at most  $C K_1^{1-\theta} K_2^\theta$ , where  $1/p_\theta = (1-\theta)/p_1 + \theta/p_2$ ,  $1/q_\theta = (1-\theta)/q_1 + \theta/q_2$ ,  $\alpha_\theta = (1-\theta)/\alpha_1 + \theta/\alpha_2$  and  $C$  only depends on  $\theta$ ,  $\alpha_i$ ,  $p_i$  and  $q_i$ . A similar property was proved by Peetre, see [P], for a extended family of spaces. Actually as J. Peetre points out — see [P, pg. 77], any interpolation theorem will do, in the sense that one can replace  $(L^{p_0}, L^{p_1})$  by an abstract pair  $(A_0, A_1)$ , and  $L^p$  by an abstract interpolation space  $A$  constructed from  $(A_0, A_1)$  and still have an inequality as before. In particular this gives us that  $\mathcal{L}^{p_\theta,\alpha_\theta}$  contains the corresponding interpolated space. The main purpose of this paper is to prove that the inclusion in the other direction does not hold. In the range  $\alpha \in (-1, n/p]$ , this was proved by Stein and Zygmund [SteZ] by constructing a linear operator bounded from  $(-\alpha)$ -Hölder continuous functions to  $(-\alpha)$ -Hölder continuous functions and from  $L^2$  to  $L^2$  which is not bounded from BMO to BMO.

Recently two of the authors, see [RV3], obtained negative results on interpolation properties in the Morrey range. To be precise, the lack of convexity which characterises interpolation functors of exponent  $\theta$ , see [BL, page 27], is proved. In that work they need the dimension  $n > 1$ .

In the present work we go further and give examples, in the one dimensional case, of operators which are bounded from  $\mathcal{L}^{p_i,\alpha} \rightarrow L^{q_i}$ ,  $i = 1, 2$ ,  $0 < \alpha \leq n/p$ ,  $p \in (0, \infty)$  and are not bounded from the intermediate  $\mathcal{L}^{p_\theta,\alpha}$  to  $L^{q_\theta}$ .

We also give a description of the predual spaces of  $\mathcal{L}^{p,\alpha}$  in the context of "block spaces" (see [SOS] and [So] for this terminology in other situations). We say that a measurable function  $b$  is a  $(q, \beta)$ -block if it is supported in a cube  $Q$  of lengthside  $r$  in such a way that

$$(1.3) \quad \left( \frac{1}{|Q|} \int_Q |b(x)|^q dx \right)^{1/q} \leq \frac{1}{r^\beta}.$$

The question of the preduals of Morrey spaces has been already studied — see [Z], and [A]. But in there the corresponding blocks have mean zero. Therefore we prove that this condition can be omitted and still have a description of the predual spaces. This question is of course related to the two possible definitions of Morrey spaces mentioned above.

In Section 2 we prove that the predual of  $\mathcal{L}^{p,\alpha}$ , when  $0 < \alpha < n/p$  is the space

$$(1.4) \quad \mathcal{B}_{q,\beta} = \left\{ \sum_{k=1}^{\infty} \lambda_k b_k : \sum |\lambda_k| < \infty \text{ and } b_k \text{ is a } (q, \beta) \text{ - block} \right\}$$

for  $\beta = n - \alpha$  and  $1/p + 1/q = 1$ .

Meanwhile the spaces defined by (1.2) reduce to  $\{0\}$  when  $\alpha > n/p$ , the definition of  $\mathcal{B}_{q,\beta}$  gives non trivial spaces in the corresponding range, beyond the preduality exponents,  $\beta > n/q$ . So it makes sense to study interpolation properties in this range; in Section 3 we also give a negative result in this context which is not covered by preduality and which is a complement to the questions posed in the 60's.

We would like to thank F. Cobos for enlightening conversations.

## 2. – Preduals

Let us start with some elementary properties of block spaces. Let us define

$$(2.1) \quad \|f\|_{\mathcal{B}_{q,\beta}} = \inf \left\{ \sum |\lambda_k| \text{ such that } f = \sum \lambda_k b_k \right\}$$

where the infimum is taken over all possible decompositions of  $f$  into  $(q, \beta)$ -blocks.

LEMMA 1.

- (a)  $\mathcal{B}_{q,\beta} \subset L^{n/\beta}$  if  $n/\beta \leq q$ .
- (b)  $L^q \subset \mathcal{B}_{q,\beta}$  if  $q \leq n/\beta$ .
- (c) For any cube  $Q$  and  $f \in L^q_{\text{loc}}$  we have

$$\|\chi_Q f\|_{\mathcal{B}_{q,\beta}} \leq |Q|^{\beta/n-1/q} \|\chi_Q f\|_q.$$

PROOF. (a) follows from Hölder and Minkowsky inequalities and the condition  $\sum |\lambda_k| < \infty$ .

(b) Denote by  $Q_k = \{x : |x_i| \leq 2^k, i = 1, \dots, n\}$ . Assume first that  $q < n/\beta$ . Then

$$f = \sum_1^{\infty} |Q_k|^{\beta/n-1/q} \|f\|_q b_k,$$

where  $b_k = \frac{f(\chi_{Q_k} - \chi_{Q_{k-1}})}{|\mathcal{Q}_k|^{\beta/n-1/q} \|f\|_q}$  is a  $(q, \beta)$ -block.

Assume now that  $q = n/\beta$  and  $f \in L^q$ . Since  $f\chi_{Q_k}$  is a Cauchy sequence in  $L^q$  then we can find  $n_k$  such that  $\|f\chi_{Q_{n_{k+1}}} - f\chi_{Q_{n_k}}\|_q < 2^{-k}$ . Now write

$$f = f\chi_{Q_{n_1}} + \sum_{k=1}^{\infty} f\chi_{Q_{n_{k+1}}} - f\chi_{Q_{n_k}}.$$

This clearly shows that  $f \in \mathcal{B}_{q,\beta}$  and  $\|f\|_{\mathcal{B}_{q,\beta}} \leq 2\|f\|_q$ .

(c) Just observe that

$$(f\chi_Q)(x) = |\mathcal{Q}|^{\beta/n-1/q} \|\chi_Q f\|_q b$$

with  $b$  a  $(q, \beta)$ -block.

REMARKS. 1.  $\mathcal{B}_{q,\beta} = L^q$  for  $\beta = n/q$ . Then its dual is  $L^p = \mathcal{L}^{p,\alpha}$  for  $\alpha = n/p$ .

2. As we observe in the introduction  $\mathcal{B}_{q,\beta}$  is a meaningful space in the case (b) of Lemma 1 and it contains  $L^q$ .

THEOREM 1. *Let  $1 < p < \infty$  and  $\alpha \in (0, n/p)$ , then if we take  $\beta$  and  $q$  such that  $\alpha + \beta = n$  and  $1/p + 1/q = 1$  we have*

$$(\mathcal{B}_{q,\beta})^* = \mathcal{L}^{p,\alpha}.$$

PROOF. Assume  $f \in \mathcal{L}^{p,\alpha}$  and take  $b$  a  $(q, \beta)$ -block supported in a cube  $\mathcal{Q}$  of side  $r$ , then

$$\begin{aligned} \left| \int_{\mathbb{R}^n} f b \right| &\leq \left( \int_{\mathcal{Q}} |f|^p \right)^{1/p} \left( \int_{\mathcal{Q}} |b|^q \right)^{1/q} \\ &= r^\alpha \left( \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} |f|^p \right)^{1/p} r^\beta \left( \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} |b|^q \right)^{1/q} \leq \|f\|_{p,\alpha}. \end{aligned}$$

Take now  $g = \sum \lambda_k b_k$ , then

$$\left| \int_{\mathbb{R}^n} f g \right| \leq \sum |\lambda_k| \left| \int_{\mathbb{R}^n} f b_k \right| \leq \sum |\lambda_k| \|f\|_{p,\alpha} \leq \|g\|_{\mathcal{B}_{q,\beta}} \|f\|_{p,\alpha}.$$

This proves that  $\mathcal{L}^{p,\alpha} \subset (\mathcal{B}_{q,\beta})^*$ .

To prove the other inclusion take  $\Phi \in (\mathcal{B}_{q,\beta})^*$  and a cube  $\mathcal{Q}$ , from (c) of Lemma 1 we have that  $\Phi$  restricted to the subset  $L^q(\mathcal{Q})$  is in  $L^q(\mathcal{Q})^*$ , and hence there exists a  $f_Q \in L^q(\mathcal{Q})$  such that

$$\int f_Q g = \Phi(g) \text{ for any } g \in L^q(\mathcal{Q}).$$

Write  $\mathbb{R}^n = \cup_1^\infty Q_k$ ,  $Q_k$  increasing, define  $f(x) = f_{Q_k}(x)$  if  $x \in Q_k$ , which makes sense since  $\int_E f_{Q_k} = \int_E f_{Q_{k+1}}$  for any Borel subset of  $Q_k$  and hence  $f_{Q_k}(x) = f_{Q_{k+1}}(x)$  a.e.  $x \in Q_k$ .

Only remains to prove that  $f \in \mathcal{L}^{p,\alpha}$ . Take a cube  $Q$  and  $j$  such that  $Q \subset Q_j$ , then

$$\begin{aligned} |Q|^{\alpha/n} \left( \frac{1}{|Q|} \int_Q |f|^p \right)^{1/p} &= |Q|^{\alpha/n-1/p} \sup_{\|h\|_q=1} \int_Q fh \, dx \\ &\leq \sup_{\|h\|_q=1} \int_{Q_j} f_{Q_j} (h \chi_Q |Q|^{\alpha/n-1/p}) \, dx \\ &\leq \|\Phi\| \|h \chi_Q |Q|^{\alpha/n-1/p}\|_{\mathcal{B}_{q,\beta}} \leq \|\Phi\|, \end{aligned}$$

since  $h \chi_Q |Q|^{\alpha/n-1/p} = h \chi_Q |Q|^{\beta/n-1/q}$  is a  $(q, \beta)$ -block.

### 3. – Interpolation

In [RV3] it was proved the lack of logarithmic convexity of the operator norm of an operator bounded from  $\mathcal{L}^{p_i,\alpha}$  to  $L^1$ ,  $i = 1, 2$  with  $1 \leq p_2 \leq \frac{n-1}{2} \leq p_1 < \infty$ ,  $0 < \alpha < n$ . This operator requires the dimension  $> 1$ . We start by exhibiting an example in dimension 1 of non-boundedness in an intermediate space. Similar examples can be constructed in higher dimension.

**THEOREM 2.** *Take  $p_1, p_2$ , and  $p_3$  such that  $1 < p_2 < p_3 < p_1 \leq 1$ ,  $\alpha = \frac{1}{p_1}$ . Then there exists  $q_1, q_2 \in (1, \infty)$  and a linear operator  $T$  such that, we have*

$$(3.1) \quad T : \mathcal{L}^{p_i,\alpha} \rightarrow L^{q_i}, \quad i = 1, 2,$$

and

$$(3.2) \quad T : \mathcal{L}^{p_3,\alpha} \not\rightarrow L^{q_3},$$

where  $\frac{1}{p_3} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$  and  $\frac{1}{q_3} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$ .

**LEMMA 2.** *Let  $0 < p_3, \alpha, \beta$  be positive numbers such that*

$$(3.3) \quad \max\{p_3, 1\} \leq \frac{\beta}{(\beta + 1)\alpha}$$

and let  $N_0$  such that  $(\beta + 1) < \frac{N_0}{\log N_0}$ . Define

$$I_j^N = [2^N + jN^\beta, 2^N + jN^\beta + 1]$$

for  $N > N_0$ ,  $N \in \mathbb{N}$ , and  $j = 0, 1, \dots, N-1$ . Then

$$\left\| \sum_{N>N_0} \sum_{j=0}^{N-1} \chi_{I_j^N} \right\|_{p_3,\alpha} \leq C,$$

where  $C$  is a universal constant.

PROOF. By inspection one can see that the biggest value of

$$|I|^\alpha \left( \frac{1}{|I|} |I \cap (\cup I_j^N)| \right)^{\frac{1}{p_3}}$$

is achieved when the  $\inf I$  is at a point  $2^N$  and  $N^{\beta+1} \approx |I|$ .

PROOF OF THEOREM 2. Choose  $\beta$  and  $q_1$  such that

$$(3.4) \quad p_3 < \frac{\beta}{\alpha(\beta+1)} < p_1,$$

$$(3.5) \quad \frac{2}{q_1} = \min \left\{ \frac{2}{p_2} + \alpha(1+\beta) - \frac{\beta}{p_1}, 2 \right\},$$

and  $q_2 = p_2$ .

Hence  $q_1 < q_2 = p_2$ .

Define  $E_N = \cup_{j=0}^{N-1} I_j^N$  with  $I_j^N$  as in Lemma 2 and

$$(3.6) \quad Tf(x) = \sum_{N>N_0} \lambda_N \chi_{E_N}(x) f(x),$$

with  $\lambda_N = \frac{1}{N^\gamma}$ ,  $\gamma$  such that

$$(3.7) \quad \frac{2}{p_2} < \gamma < \frac{2}{q_3} = \frac{2(1-\theta)}{q_1} + \frac{2\theta}{q_2}.$$

Notice that from (3.5)  $\frac{1}{q_3} > \frac{1}{p_2}$ . Hence we trivially have

$$(3.8) \quad \begin{aligned} \|Tf\|_{q_1} &= \left( \sum_N \lambda_N^{q_1} \int_{E_N} |f|^{q_1} \right)^{\frac{1}{q_1}} \\ &\leq \left( \sum_N \lambda_N^{q_1} N^{q_1 \left( \frac{1}{q_1} - \frac{1}{p_1} \right)} \|f\|_{L^{p_1}(E_N)}^{q_1} \right)^{\frac{1}{q_1}} \\ &\leq \left( \sum_N \lambda_N^{q_1} N^{1 - q_1 \left( \alpha(1+\beta) - \frac{\beta}{p_1} \right)} \right)^{\frac{1}{q_1}} \|f\|_{p_1, \alpha}, \end{aligned}$$

since

$$\|f\|_{L^{p_1}(E_N)} \leq N^{\frac{(\beta+1)}{p_1} - \alpha(\beta+1)} \|f\|_{p_1, \alpha}.$$

Then  $\|Tf\|_{q_1} \leq C \|f\|_{p_1, \alpha}$ , follows from (3.5) and (3.7).

We have from (3.7) that  $1 - \gamma q_2 < -1$ , and for  $q_2 = p_2$ :

$$\begin{aligned} \|Tf\|_{q_2} &\leq \left( \sum_N \lambda_N^{q_2} \sum_{j=1}^{N-1} \|f\|_{L^{p_2}(I_j^N)}^{p_2} \right)^{1/p_2} \\ &\leq \left( \sum_N \lambda_N^{q_2} N \right)^{\frac{1}{q_2}} \|f\|_{p_2, \alpha} \leq C \|f\|_{p_2, \alpha}. \end{aligned}$$

On the other hand we know from Lemma 2 that  $f = \sum_N \chi_{E_N} \in \mathcal{L}^{p_3, \alpha}$  and from (3.7)  $1 - \gamma q_3 > -1$ , hence

$$\|Tf\|_{q_3} = \left( \sum_N \lambda_N^{q_3} N \right)^{\frac{1}{q_3}} = \left( \sum N^{1-\gamma q_3} \right)^{\frac{1}{q_3}} = \infty.$$

The proof is over.

The next theorem states that interpolation for the blocks spaces  $\mathcal{B}_{q, \beta}$  does not hold between points at both sides of the line  $q = \frac{1}{\beta}$ . In fact we give an operator bounded  $L^{p_i} \rightarrow \mathcal{B}_{q_i, \beta}$ ,  $i = 1, 2$  which is not bounded from  $L^{p_\theta} \rightarrow \mathcal{B}_{q_\theta, \beta}$ , and such that  $\mathcal{B}_{q_2, \beta}$  is on the predual range and  $\mathcal{B}_{q_1, \beta}$  is out of it.

**THEOREM 3.** *Let  $1 \leq q_1 < q_2 \leq 2$ . There exist  $\beta \in (\frac{1}{q_2}, \frac{1}{q_1})$ ,  $\theta \in (0, 1)$ ,  $p_1, p_2$  and a linear operator  $T$  such that we have*

$$(3.9) \quad T : L^{p_1} \rightarrow \mathcal{B}_{q_1, \beta}$$

$$(3.10) \quad T : L^{p_2} \rightarrow \mathcal{B}_{q_2, \beta}$$

and

$$(3.11) \quad T : L^{p_\theta} \not\rightarrow \mathcal{B}_{q_\theta, \beta}$$

where  $\frac{1}{p_\theta} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$  and  $\frac{1}{q_\theta} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$ .

In the proof of Theorem 3 the following lemma will be used:

**LEMMA 3.** *Let  $\{E_k\} \subset \mathbb{R}$  and let  $B_k = \text{co}(E_k)$  denotes its convex hull, let  $q_1 < p_1$ ,  $\beta > 0$  and  $f \in L^{p_1}(\mathbb{R})$  and  $\|f\|_{p_1} = 1$ . If  $\{B_k\}$  are disjoint, and*

$$(3.12) \quad \lambda_k |E_k|^{\frac{1}{q_1} - \frac{1}{p_1}} |B_k|^{\beta - \frac{1}{q_1}} \in l^1,$$

then  $\sum \lambda_k f \chi_{E_k} \in \mathcal{B}_{q_1, \beta}$ .

**PROOF.** Just observe that from Hölder  $|E_k|^{\frac{1}{p_1} - \frac{1}{q_1}} |B_k|^{\frac{1}{q_1} - \beta} f \chi_{E_k}$  is a  $(q_1, \beta)$ -block.



PROOF OF THEOREM 3. Observe that  $\frac{1}{q'_2} < \frac{1}{q_1}$ . Take  $p_2 = q_2$  and  $p_2 < p_1$  such that

$$(3.13) \quad \frac{1}{q'_2} < \frac{1}{q_1} - \frac{1}{p_1}.$$

Now choose  $\theta$  such that

$$(3.14) \quad \frac{1}{q'_2} < (1 - \theta) \left( \frac{1}{q_1} - \frac{1}{p_1} \right),$$

and now  $\beta$  such that

$$(3.15) \quad \frac{1}{q_\theta} < \beta < \frac{1}{q_1}$$

where  $\frac{1}{q_\theta} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$ .

Consider now  $E_k = [2^k, 2^k + 1) \cup [2^{k+1}, 1, 2^{k+1})$  and define

$$T(f) = \left( \sum_{k=1}^{\infty} k^{-\gamma} \chi_{E_k} \right) f$$

for  $\gamma$  chosen such that

$$(3.16) \quad \frac{1}{p'_2} = \frac{1}{q'_2} < \gamma < (1 - \theta) \left( \frac{1}{q_1} - \frac{1}{p_1} \right).$$

Since  $\beta < \frac{1}{q_1}$ ,  $|E_k| = 2$  and  $|B_k| = 2^k$  then Lemma 3 implies (3.9).

Now from condition (3.15) we have that  $\frac{1}{q_2} < \beta$  what allows us to use Theorem 1 and write (remember that  $q_2 = p_2$ ) for  $\alpha = 1 - \beta$ :

$$\begin{aligned} \left\| \sum k^{-\gamma} f \chi_{E_k} \right\|_{B_{q_2, \beta}} &= \sup_{\|g\|_{q'_2, \alpha} \leq 1} \int \left( \sum k^{-\gamma} f \chi_{E_k} \right) g \\ &= \sup_{\|g\|_{p'_2, \alpha} \leq 1} \int \left( \sum k^{-\gamma} g \chi_{E_k} \right) f \\ &\leq \|f\|_{p_2} \sup_{\|g\|_{p'_2, \alpha} \leq 1} \left\| \sum k^{-\gamma} g \chi_{E_k} \right\|_{p'_2} \\ &= \|f\|_{p_2} \sup_{\|g\|_{p'_2, \alpha} \leq 1} \left( \sum k^{-\gamma p'_2} \int_{E_k} |g|^{p'_2} \right)^{\frac{1}{p'_2}} \\ &\leq \|f\|_{p_2} 2^{\frac{1}{p'_2}} \left( \sum k^{-\gamma p'_2} \right)^{\frac{1}{p'_2}}. \end{aligned}$$

The last inequality follows from the fact that  $\int_{E_k} |g|^{p'_2} \leq 2 \|g\|_{p'_2, \alpha}^{p'_2}$ . The above series converges from (3.16). This proves (3.10).

Finally, since  $\frac{1}{q_\theta} < \beta$ , then (3.11) is equivalent to see that  $T^* = T$  is not bounded from  $\mathcal{L}^{q'_\theta, 1-\beta}$  to  $L^{p'_\theta}$ .

Take now  $A_N = \cup_{k=1}^N E_k$  and  $f_N = |A_N|^{-\frac{1}{q_\theta}} \chi_{A_N}$ .

It is elementary to see that  $\|f_N\|_{q'_\theta, 1-\beta} \leq 1$ .

On the other hand

$$\|T(f_N)\|_{p'_\theta} = 2^{\frac{1}{p'_\theta}} (2N)^{-\frac{1}{q'_\theta}} \left( \sum_{k=1}^N k^{-\gamma p'_\theta} \right)^{\frac{1}{p'_\theta}}.$$

Note that

$$(3.17) \quad \frac{1}{q_\theta} - \frac{1}{p_\theta} = (1 - \theta) \left( \frac{1}{q_1} - \frac{1}{p_1} \right)$$

and then (3.16) gives that  $\gamma p'_\theta < 1$  what allows to write

$$(3.18) \quad \sum_{k=1}^N k^{-\gamma p'_\theta} \geq CN^{-\gamma p'_\theta + 1}.$$

Using (3.17) and (3.18) we get that

$$\|T(f_N)\|_{p'_\theta} \geq CN^{-\gamma + (1-\theta)(\frac{1}{q_1} - \frac{1}{p_1})}.$$

Now (3.16) gives that  $\sup_N \|T(f_N)\|_{p'_\theta} = \infty$  and the proof is completed.

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